QUINTIC AND SEPTIC EISENSTEIN SERIES

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ABSTRACT. Using results that were well–known to Ramanujan, we give proofs of some results for Eisenstein series in the lost notebook. Our proofs have the additional advantage that it is not necessary to know the results in advance; that is, the proofs are derivations as opposed to verifications.

1. Introduction

Ramanujan's Eisenstein series are defined by

$$P(q) = 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \quad Q(q) = 1 + 240\sum_{j=1}^{\infty} \frac{j^3q^j}{1-q^j}$$

and

$$R(q) = 1 - 504\sum_{j=1}^{\infty} \frac{j^5q^j}{1-q^j},$$

where $|q| < 1$. Let

$$(q)_\infty = \prod_{j=1}^{\infty}(1-q^j).$$

On pages 50 and 51 of the lost notebook [20], Ramanujan recorded the result

$$R(q^5) = \frac{(q)_\infty^{15}}{(q^5)_\infty^3}\left(1 + 4q\frac{(q^5)_\infty^6}{(q)_\infty^6} - q^2\frac{(q^5)_\infty^{12}}{(q)_\infty^{12}}\right)\sqrt{1 + 22q\frac{(q^5)_\infty^6}{(q)_\infty^6} + 125q^2\frac{(q^5)_\infty^{12}}{(q)_\infty^{12}}},$$

as well as analogous formulas involving $Q(q)$, $Q(q^5)$ and $R(q)$. He also gave formulas involving the products $(q)_\infty$ and $(q^7)_\infty$ for the Eisenstein series $Q(q)$, $Q(q^7)$, $R(q)$ and $R(q^7)$ on page 53. The goal of this work is to prove all of these formulas — see Theorems 2.6 and 3.4 — using results that were well–known to Ramanujan.

The first proof of these results was given in 1989 by S. Raghavan and S. S. Rangachari [16]. Their proof relied on the theory of modular forms, a method not familiar to Ramanujan.

In 2000, proofs were given by B. C. Berndt et al [4]. These proofs have the advantage of depending only on modular equations that are in Ramanujan's notebooks. However, the authors conceded that "*... some of our algebraic*

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manipulations are rather laborious, and we resorted at times to (using computer algebra). It is therefore clear to us that Ramanujan's calculations, at least in some cases, were more elegant than ours."

In 2004, a proof using the theory of elliptic functions was given by Z.-G. Liu [15]. Although this only requires the elementary theory of functions of a complex variable, such as the residue theorem, considerable skill is needed to construct the elliptic functions that lead to the identities. Moreover, it is widely believed that Ramanujan did not use tools such as the residue theorem in his work. He undoubtedly possessed another method.

In this work, we will give proofs that use results that were well–known to Ramanujan. Our proof has the additional advantage that it is a derivation, not just a verification. Ramanujan could easily have performed the calculations in our proof. Although we will never know Ramanujan's method, it is plausible that he may have been led to the formulas that we will prove in Theorems 2.6 and 3.4 by using the techniques that we present.

This work is organized as follows. The quintic identities are analyzed in Section 2 and the septic identities in Section 3. Each section begins by recounting some identities that were well–known to Ramanujan. The key to our method lies in using the Ramanujan differential equations to construct a system of four non-linear equations which turn out to have polynomial solutions. Each section ends with a second order linear differential equation.

2. Quintic Identities

Let

$$
\begin{aligned}
X &= q\prod_{j=1}^{\infty}\frac{(1-q^{5j})^6}{(1-q^j)^6},\\
Z &= \prod_{j=1}^{\infty}\frac{(1-q^{j})^5}{(1-q^{5j})},\\
\Lambda &= q\prod_{j=1}^{\infty}\frac{(1-q^{5j-4})^5(1-q^{5j-1})^5}{(1-q^{5j-3})^5(1-q^{5j-2})^5},\\
A &= (1+22X+125X^2)^{1/2}.
\end{aligned}
$$

Lemma 2.1.

$$
Z = 1-5\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j},
$$

where $\left(\frac{j}{5}\right)$ *is the Legendre symbol.*

Lemma 2.1 was given by Ramanujan in his second notebook [19, Chapter 19, Entry 9 (v)], and also in the lost notebook [20, pp. 354, 357]. For proofs, see Berndt's book [3, pp. 257–261], or the papers by J. M. Dobbie [8] and M. Hirschhorn [10]. References to other proofs are given in [3] and [10].

Lemma 2.2.

$$\frac{1}{\Lambda} - 11 - \Lambda = \frac{1}{X}.$$

Lemma 2.2 was given by Ramanujan in his paper [18]. A simple proof using only the Jacobi triple product identity has been given by Hirschhorn [11]. More information, and references to other proofs, can be found in the book by G. E. Andrews and B. C. Berndt [2, pp. 11–12]. The function Λ is the fifth power of the Rogers-Ramanujan continued fraction.

Lemma 2.3.

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2},$$

and

$$Q^3 - R^2 = 1728q\prod_{j=1}^{\infty}(1 - q^j)^{24}.$$

The first three results in Lemma 2.3 are called Ramanujan's differential equations, and the last result is a consequence of them. All of the results in Lemma 2.3 were recorded by Ramanujan in his second notebook [19, Chapter 14, Entries 12, 13]. Ramanujan's proof appears in his paper [17]. A different proof, relying on the Jacobi and quintuple product identities, has been given by H. H. Chan [5].

Theorem 2.4.

$$\begin{aligned} q\frac{d}{dq}\log\Lambda &= Z, \\ q\frac{d}{dq}\log X &= \frac{1}{4}\left(5P(q^5) - P(q)\right) = AZ, \\ AX\frac{dZ}{dX} &= \frac{5}{24}\left(P(q) - P(q^5)\right). \end{aligned}$$

Proof. The first result is a trivial consequence of Lemma 2.1.

The first equality in the second result follows immediately from the definitions of X and P. The second equality in the second result may be proved as follows. By Lemmas 2.1 and 2.2, we find that

$$q\frac{d}{dq}\log X = -X\,q\frac{d}{dq}\left(\frac{1}{\Lambda} - 11 - \Lambda\right) = X\left(\frac{1}{\Lambda^2} + 1\right)\,q\frac{d\Lambda}{dq}.$$

Now apply the first part of Theorem 2.4 and Lemma 2.2, to obtain

$$\begin{aligned} q\frac{d}{dq}\log X &= ZX\left(\frac{1}{\Lambda}+\Lambda\right) \\ &= ZX\sqrt{\left(\frac{1}{\Lambda}-\Lambda\right)^2+4} \\ &= ZX\sqrt{\left(\frac{1}{X}+11\right)^2+4} \\ &= AZ. \end{aligned}$$

It remains to prove the third result. Observe that from the second part of Theorem 2.4 we have

$$\frac{dX}{dq}=\frac{AXZ}{q}>0 \quad \text{for } -1<q<1.$$

Therefore the inverse function $q=q(X)$ exists. Consequently the function Z, which was defined in terms of q, may be regarded as a function of X and the derivative dZ/dX may be computed by the chain rule. From the definitions of Z and P we have

$$q\frac{d}{dq}\log Z=\frac{5}{24}\left(P(q)-P(q^5)\right).$$

Therefore

$$\frac{dZ}{dX}=q\frac{dZ}{dq}\Big/ q\frac{dX}{dq}=\frac{5Z}{24}\frac{\left(P(q)-P(q^5)\right)}{AXZ}.$$

This completes the proof. □

Remark. The second equality in the second part of Theorem 2.4 is due to Ramanujan [19, Chapter 21, Entry 4 (i)]. A proof was given by Berndt [3, pp. 464–465]. The proof we have given above is the same as the proof by R. Lewis and Z.-G. Liu [13].

Theorem 2.5.

$$\begin{aligned} 25Q(q^5)-Q(q) &= 24Z^2(1-125X^2), \\ 125R(q^5)-R(q) &= 18000AX^2Z^3+AZ\left(125Q(q^5)-Q(q)\right). \end{aligned}$$

Proof. Applying $q\frac{d}{dq}$ to the second result in Theorem 2.4 we get

$$12q\frac{d}{dq}(5P(q^5)-P(q))=48\left(A\frac{dZ}{dX}+(11+125X)\frac{Z}{A}\right)q\frac{dX}{dq}.$$

The derivatives may be computed using Lemma 2.3 and Theorem 2.4. The result is

$$\begin{aligned} &25P(q^5)^2-P(q)^2-25Q(q^5)+Q(q) \\ &= 48\left(\frac{5}{24X}\left(P(q)-P(q^5)\right)+(11+125X)\frac{Z}{A}\right)AXZ. \end{aligned}$$

Completing the squares for $P(q)$ and $P(q^5)$ we obtain

$$25Q(q^5) - Q(q) = \left(5P(q^5) + AZ\right)^2 - (P(q) + 5AZ)^2 + 24Z^2(1 - 125X^2).$$

By the second part of Theorem 2.4, this simplifies to

$$25Q(q^5) - Q(q) = 24Z^2(1 - 125X^2).$$

This proves the first part of the theorem. Applying $q\frac{d}{dq}$ again, we get

$$3q\frac{d}{dq}\left(25Q(q^5) - Q(q)\right) = \left(144Z\frac{dZ}{dX}(1 - 125X^2) - 18000Z^2X\right) q\frac{dX}{dq}.$$

The derivatives may be computed using Lemma 2.3 and Theorem 2.4. The result simplifies to

$$\begin{aligned} &125P(q^5)Q(q^5) - P(q)Q(q) - 125R(q^5) + R(q) \\ &= \; 30Z^2(1 - 125X^2)\left(P(q) - P(q^5)\right) - 18000AX^2Z^3. \end{aligned}$$

By the first part of this theorem, this is equivalent to

$$\begin{aligned} &125P(q^5)Q(q^5) - P(q)Q(q) - 125R(q^5) + R(q) \\ &= \; \frac{5}{4}\left(25Q(q^5) - Q(q)\right)\left(P(q) - P(q^5)\right) - 18000AX^2Z^3. \end{aligned}$$

Rearranging, we find

$$125R(q^5) - R(q) = 18000AX^2Z^3 + \frac{1}{4}\left(5P(q^5) - P(q)\right)\left(125Q(q^5) - Q(q)\right).$$

Finally, using the second part of Theorem 2.4, we complete the proof. □

Theorem 2.6.

$$\begin{aligned} Q(q) &= Z^2(1 + 250X + 3125X^2), \\ Q(q^5) &= Z^2(1 + 10X + 5X^2), \\ R(q) &= Z^3A(1 - 500X - 15625X^2), \\ R(q^5) &= Z^3A(1 + 4X - X^2). \end{aligned}$$

Proof. From the last part of Lemma 2.3 and the definitions of Z and X we have

$$Q(q)^3 - R(q)^2 = 1728Z^6X, \tag{2.1}$$

$$Q(q^5)^3 - R(q^5)^2 = 1728Z^6X^5. \tag{2.2}$$

Let

$$u_1 = \frac{Q(q)}{Z^2}, \quad u_2 = \frac{Q(q^5)}{Z^2}, \quad u_3 = \frac{R(q)}{AZ^3}, \quad u_4 = \frac{R(q^5)}{AZ^3}.$$

Observe that when $q = 0$ we have $X = 0$, $Z = 1$ and $u_1 = u_2 = u_3 = u_4 = 1$. The results of Theorem 2.5 and (2.1) and (2.2) imply

$$-u_1 + 25u_2 = 24(1 - 125X^2), \tag{2.3}$$

$$u_1 - 125u_2 - u_3 + 125u_4 = 18000X^2, \tag{2.4}$$

$$u_1^3 - (1 + 22X + 125X^2)u_3^2 = 1728X, \tag{2.5}$$

$$u_2^3 - (1 + 22X + 125X^2)u_4^2 = 1728X^5. \tag{2.6}$$

We seek a solution of (2.3)–(2.6) that expresses each of u_1, u_2, u_3 and u_4 as power series in X. We begin by determining the linear approximations, and seek solutions of the form

$$u_j = 1 + a_j X + O(X^2), \quad 1 \le j \le 4. \tag{2.7}$$

If we substitute (2.7) into (2.3)–(2.6) and compare the first order terms, i.e., the coefficients of X, we obtain the linear system

$$\begin{pmatrix} -1 & 25 & 0 & 0 \\ 1 & -125 & -1 & 125 \\ 3 & 0 & -2 & 0 \\ 0 & 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1750 \\ 22 \end{pmatrix}.$$

Therefore, $a_1 = 250$, $a_2 = 10$, $a_3 = -500$, $a_4 = 4$. Next, we determine the quadratic approximations, and set

$$u_j = 1 + a_j X + b_j X^2 + O(X^3), \quad 1 \le j \le 4. \tag{2.8}$$

If we substitute (2.8) into (2.3)–(2.6) and compare coefficients of X^2, we obtain the linear system

$$\begin{pmatrix} -1 & 25 & 0 & 0 \\ 1 & -125 & -1 & 125 \\ 3 & 0 & -2 & 0 \\ 0 & 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} -3000 \\ 18000 \\ 40625 \\ 17 \end{pmatrix}.$$

Therefore, $b_1 = 3125$, $b_2 = 5$, $b_3 = -15625$, $b_4 = -1$.

It turns out that the cubic, quartic and higher approximations are identical to the quadratic approximations. This is because the quadratic functions

$$\begin{aligned} u_1 &= 1 + 250X + 3125X^2, \\ u_2 &= 1 + 10X + 5X^2, \\ u_3 &= 1 - 500X - 15625X^2, \\ u_4 &= 1 + 4X - X^2 \end{aligned}$$

are in fact solutions of (2.3)–(2.6), as can be checked by direct substitution. The implicit function theorem implies this is the unique solution of (2.3)–(2.6) that satisfies $u_1 = u_2 = u_3 = u_4 = 1$ when $X = 0$. To see this, write (2.3)–(2.6) as a vector equation $\mathbf{F}(X, \mathbf{u}) = \mathbf{g}(X)$, and observe that the Jacobian determinant is

$$\left| \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \right|_{X=0} = \begin{vmatrix} -1 & 25 & 0 & 0 \\ 1 & -125 & -1 & 125 \\ 3 & 0 & -2 & 0 \\ 0 & 3 & 0 & -2 \end{vmatrix} = -200 \neq 0.$$

This completes the proof. □

The modular j-function is defined for $\operatorname{Im}(\tau) > 0$ by

$$j(\tau) = \frac{1728 Q(q)^3}{Q(q)^3 - R(q)^2}, \quad \text{where} \quad q = \exp(2\pi i \tau).$$

The next result gives formulas for the j-function in terms of Λ.

Theorem 2.7.

$$\begin{aligned} j(\tau) &= \frac{(1+228\Lambda+494\Lambda^2-228\Lambda^3+\Lambda^4)^3}{\Lambda(1-11\Lambda-\Lambda^2)^5}, \\ j(5\tau) &= \frac{(1-12\Lambda+14\Lambda^2+12\Lambda^3+\Lambda^2)^4}{\Lambda^5(1-11\Lambda-\Lambda^2)}. \end{aligned}$$

Proof. Use Theorem 2.6 to write $j(\tau)$ and $j(5\tau)$ as rational functions of X, then use Lemma 2.2 to express X in terms of Λ. □

Remark. The first equation in Theorem 2.7, with r^5 written in place of Λ, is called the the icosahedral equation. Details of the connection with the icosahedron are in the paper by W. Duke [9].

The next result is a second order differential equation for Z in terms of X. Four equivalent formulations will be given in Corollary 2.9, and an application to divisor sums will be given in Section 4.

Theorem 2.8.

$$AXZ\frac{d}{dX}\left(AX\frac{dZ}{dX}\right) = \frac{1}{2}\left(AX\frac{dZ}{dX}\right)^2 - \frac{5}{2}X(2+25X)Z^2.$$

Proof. Applying $q\frac{d}{dq}$ to the third result in Theorem 2.4 we get

$$\frac{d}{dX}\left(AX\frac{dZ}{dX}\right)q\frac{dX}{dq} = \frac{5}{24}q\frac{d}{dq}\left(P(q)-P(q^5)\right).$$

Use Lemma 2.3 and Theorem 2.4 to compute the derivatives. The result is

$$AXZ\frac{d}{dX}\left(AX\frac{dZ}{dX}\right) = \frac{5}{288}\left(P(q)^2-5P(q^5)^2-Q(q)+5Q(q^5)\right). \tag{2.9}$$

From Theorem 2.4 we have

$$P(q) = AZ+6AX\frac{dZ}{dX} \quad \text{and} \quad P(q^5) = AZ+\frac{6}{5}AX\frac{dZ}{dX}. \tag{2.10}$$

Using these and Theorem 2.5 in (2.9), we obtain

$$\begin{aligned} &AXZ\frac{d}{dX}\left(AX\frac{dZ}{dX}\right) \\ &= \frac{5}{288}\left(-4A^2Z^2+\frac{144}{5}\left(AX\frac{dZ}{dX}\right)^2+Z^2(4-200X-3100X^2)\right). \end{aligned}$$

Since $A^2 = 1+22X+125X^2$, this completes the proof. □

Corollary 2.9. *Let*

$$Z = U^2 \qquad \textit{and} \qquad F(X) = \frac{-AX}{5Z}\frac{dZ}{dX}.$$

Then

$$AX\frac{d}{dX}\left(AX\frac{dU}{dX}\right)+\frac{5}{4}X(2+25X)U=0, \tag{2.11}$$

$$\frac{d}{d\Lambda}\left(\Lambda\frac{dU}{d\Lambda}\right)=\frac{5}{4}\frac{(2\Lambda+1)(\Lambda-2)}{(\Lambda^2+11\Lambda-1)^2}U, \tag{2.12}$$

$$\begin{aligned} &q\frac{d}{dq}\left(P(q)-P(q^5)\right) \\ &\quad= \frac{5}{48}\left(P(q)-P(q^5)\right)^2-\frac{5}{52}\left(Q(q)-Q(q^5)\right) \\ &\qquad -\frac{12}{13}q(q)_\infty^4(q^5)_\infty^4. \end{aligned} \tag{2.13}$$

Furthermore, we have

$$P(q) = Z\left(A-30F(X)\right), \tag{2.14}$$

$$P(q^5) = Z\left(A-6F(X)\right), \tag{2.15}$$

and

$$A\frac{dF}{dX}=1+\frac{25}{2}X+\frac{5}{2X}\left(F(X)\right)^2. \tag{2.16}$$

Proof. The result (2.11) follows by making the change of variable $U^2=Z$, and (2.12) follows from this by making the further change of variable from X to Λ given by Lemma 2.2. We suppress the details of these routine calculations.

Let us prove (2.13). The second part of Theorem 2.4 may be rewritten in terms of differential operators as

$$AXZ\frac{d}{dX}=q\frac{d}{dq}.$$

Using this, together with the third part of Theorem 2.4, we see that Theorem 2.8 is equivalent to

$$\frac{5}{24}q\frac{d}{dq}\left(P(q)-P(q^5)\right)=\frac{25}{1152}\left(P(q)-P(q^5)\right)^2-\frac{5}{2}X(2+25X)Z^2. \tag{2.17}$$

Next, by Theorem 2.6 we have

$$\begin{aligned} X(2+25X)Z^2 &= \frac{5}{624}\left(Q(q)-Q(q^5)\right)+\frac{1}{13}Z^2X \\ &= \frac{5}{624}\left(Q(q)-Q(q^5)\right)+\frac{1}{13}q(q)_\infty^4(q^5)_\infty^4. \end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.17) we obtain (2.13).

The results in (2.14) and (2.15) are just restatements of the results in (2.10). It remains to prove (2.16). Applying the product rule to the definition of $F(X)$, we get

$$\frac{dF}{dX}=\frac{-1}{5Z}\frac{d}{dX}\left(AX\frac{dZ}{dX}\right)+\frac{AX}{5Z^2}\left(\frac{dZ}{dX}\right)^2.$$

By Theorem 2.8, this is equivalent to

$$\frac{dF}{dX} = \frac{-1}{5AXZ^2}\left\{\frac{1}{2}\left(AX\frac{dZ}{dX}\right)^2 - \frac{5}{2}X(2+25X)Z^2\right\} + \frac{AX}{5Z^2}\left(\frac{dZ}{dX}\right)^2.$$

Multiplying by A and simplifying, we get

$$\begin{aligned} A\frac{dF}{dX} &= \frac{1}{10X}\left(\frac{AX}{Z}\frac{dZ}{dX}\right)^2 + 1 + \frac{25}{2}X \\ &= \frac{5}{2X}\left(F(X)\right)^2 + 1 + \frac{25}{2}X. \end{aligned}$$

□

Remarks. The differential equations (2.11) and (2.12) are both linear, whereas the differential equation in Theorem 2.8 is nonlinear. Equation (2.13) is due to Raghavan and Rangachari [16, (53)]. It is an analogue of the first equation in Lemma 2.3 for $\Gamma_0(5)$. Equations (2.14) to (2.16) were given by Ramanujan in the lost notebook [20, p. 44].

3. Septic Identities

For this section, let

$$\begin{aligned} x &= q\prod_{j=1}^{\infty}\frac{(1-q^{7j})^4}{(1-q^j)^4}, \\ z &= \prod_{j=1}^{\infty}\frac{(1-q^j)^7}{(1-q^{7j})}, \\ \sigma &= 1 + 2\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{q^j}{1-q^j}. \end{aligned}$$

Lemma 3.1.

$$\begin{aligned} \sigma^2 &= \frac{1}{6}(7P(q^7) - P(q)), \\ \sigma^3 &= z(1 + 13x + 49x^2). \end{aligned}$$

Both of the results in Lemma 3.1 were given by Ramanujan in his second notebook [19, Chapter 21, Entry 5 (i)]. Berndt [3, pp. 467–473] proved these results using modular equations of order 7.

Other proofs of the first result in Lemma 3.1 have been given by H. H. Chan and Y. L. Ong [7], Liu [14] and K. S. Williams [23]. We would like to add that the first result in Lemma 3.1 is a special case of the elliptic function identity

$$(\zeta(a) + \zeta(b) + \zeta(c))^2 = \wp(a) + \wp(b) + \wp(c), \quad \text{if } a + b + c = 0. \tag{3.1}$$

Here $\wp$ is the Weierstrass elliptic function and ζ is the corresponding Weierstrass zeta function. If we take the periods of the Weierstrass elliptic functions to be 2π and $2\pi\tau$ where $\mathrm{Im}(\tau) > 0$ and let $q = e^{2\pi i\tau}$, then we have the series expansions (cf. [22, p. 460, Ex. 35]):

$$\begin{aligned}\zeta(\theta) &= \frac{1}{2}\cot\frac{\theta}{2} + \frac{\theta P(q)}{12} + 2\sum_{j=1}^{\infty}\frac{q^j}{1-q^j}\sin j\theta,\\ \wp(\theta) &= \frac{1}{4}\csc^2\frac{\theta}{2} - \frac{P(q)}{12} - 2\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}\cos j\theta,\end{aligned}$$

valid for $-2\pi\,\mathrm{Im}(\tau) < \mathrm{Im}(\theta) < 2\pi\,\mathrm{Im}(\tau)$. Taking $(a,b,c) = (2\pi/7, 4\pi/7, -6\pi/7)$ in (3.1), we get the first result in Lemma 3.1. The formula (3.1) may be found in [22, p. 446], where it was attributed to Frobenius and Stickelberger.

Other proofs of the second result in Lemma 3.1 were given by Chan and Ong [7] and Liu [14].

The function $\sigma(q)$ also has the representation

$$\sigma(q) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{j^2+jk+2k^2},$$

although we will not require this. This follows from a result of Dirichlet. See [12, Theorem 204].

Theorem 3.2.

$$\begin{aligned}q\frac{d}{dq}\log x &= \frac{1}{6}\left(7P(q^7) - P(q)\right) = \sigma^2,\\ q\frac{d}{dq}\log z &= \frac{7}{24}\left(P(q) - P(q^7)\right),\\ 3q\frac{d\sigma}{dq} &= \frac{7\sigma}{24}\left(P(q) - P(q^7)\right) + zx(13+98x),\\ \frac{\sigma^2 x}{z}\frac{dz}{dx} &= \frac{7}{24}\left(P(q) - P(q^7)\right).\end{aligned}$$

Proof. The first equality in the first result follows immediately from the definitions of x and P. The second equality in the first result follows from Lemma 3.1. The second result is a direct consequence of the definitions of z and P.

Let us prove the third result. Applying $q\frac{d}{dq}$ to the second result in Lemma 3.1 gives

$$3\sigma^2\, q\frac{d\sigma}{dq} = q\frac{dz}{dq}(1+13x+49x^2) + zq\frac{dx}{dq}(13+98x).$$

The first two parts of this theorem may be used to evaluate the derivatives on the right. The result is

$$3\sigma^2\, q\frac{d\sigma}{dq} = \frac{7z}{24}(1+13x+49x^2)\left(P(q) - P(q^7)\right) + \frac{zx}{6}(13+98x)\left(7P(q^7) - P(q)\right).$$

Applying Lemma 3.1 to the right hand side we get

$$3\sigma^2\, q\frac{d\sigma}{dq} = \frac{7\sigma^3}{24}\left(P(q) - P(q^7)\right) + z\sigma^2 x(13+98x).$$

This proves the third result.

By Lemma 3.1 and the first part of this theorem we have

$$\frac{dx}{dq} = \frac{x}{6q}\left(7P(q^7) - P(q)\right) = \frac{x\sigma^2}{q} > 0 \quad \text{for } -1 < q < 1.$$

Therefore the inverse function $q = q(x)$ exists, and the function z may be regarded as a function of x. By Lemma 3.1 and the first two parts of Theorem 3.2 we have

$$\begin{aligned} \frac{dz}{dx} &= \left(q\frac{dz}{dq}\right)\Big/\left(q\frac{dx}{dq}\right) \\ &= \frac{7z}{4x}\frac{\left(P(q) - P(q^7)\right)}{(7P(q^7) - P(q))} \\ &= \frac{7z}{24\sigma^2 x}\left(P(q) - P(q^7)\right). \end{aligned}$$

This completes the proof of the theorem. □

Theorem 3.3.

$$\begin{aligned} 49Q(q^7) - Q(q) &= 48\sigma z(1 - 49x^2), \\ 343R(q^7) - R(q) &= \sigma^2\left(343Q(q^7) - Q(q)\right) \\ &\qquad + 48z^2 x(-13 + 196x + 4459x^2 + 19208x^3). \end{aligned}$$

Proof. Applying $q\frac{d}{dq}$ to the first part of Lemma 3.1 we get

$$12q\frac{d}{dq}\left(7P(q^7) - P(q)\right) = 144\sigma q\frac{d\sigma}{dq}.$$

The derivatives may be computed using Lemma 2.3 and Theorem 3.2, giving

$$\begin{aligned} &49P(q^7)^2 - P(q)^2 - 49Q(q^7) + Q(q) \\ &= 14\sigma^2\left(P(q) - P(q^7)\right) + 48zx\sigma(13+98x). \end{aligned}$$

Completing the squares for $P(q)$ and $P(q^7)$ we obtain

$$\begin{aligned} &49Q(q^7) - Q(q) \\ &= \left(7P(q^7) + \sigma^2\right)^2 - \left(P(q) + 7\sigma^2\right)^2 + 48\sigma^4 - 48zx\sigma(13+98x). \end{aligned}$$

Applying both parts of Lemma 3.1, this simplifies to

$$\begin{aligned} 49Q(q^7) - Q(q) &= 48\sigma z(1 + 13x + 49x^2) - 48zx\sigma(13+98x) \\ &= 48\sigma z(1 - 49x^2). \end{aligned}$$

This proves the first result. Applying $q\frac{d}{dq}$ again, we get

$$3q\frac{d}{dq}\left(49Q(q^7) - Q(q)\right) = 144q\frac{d(\sigma z)}{dq}(1 - 49x^2) - 14122\sigma zxq\frac{dx}{dq}. \tag{3.2}$$

From Lemma 3.1 and Theorem 3.2 we find that

$$q\frac{dx}{dq} = \sigma^2 x \quad \text{and} \quad q\frac{d(\sigma z)}{dq} = \frac{7\sigma z}{18}(P(q) - P(q^7)) + \frac{z^2 x}{3}(13 + 98x).$$

Using these and Lemma 2.3 in (3.2), we get

$$\begin{aligned} &343P(q^7)Q(q^7) - P(q)Q(q) - 343R(q^7) + R(q) \\ &= 56\sigma z(P(q) - P(q^7))(1 - 49x^2) \\ &\quad +48z^2 x(13 + 98x)(1 - 49x^2) - 14112\sigma^3 z x^2. \end{aligned}$$

By the first part of this theorem and the second part of Lemma 3.1, this is equivalent to

$$\begin{aligned} &343P(q^7)Q(q^7) - P(q)Q(q) - 343R(q^7) + R(q) \\ &= \frac{7}{6}(P(q) - P(q^7))(49Q(q^7) - Q(q)) \\ &\quad +48z^2 x(13 + 98x)(1 - 49x^2) - 14112z^2 x^2(1 + 13x + 49x^2). \end{aligned}$$

Rearranging, we get

$$\begin{aligned} 343R(q^7) - R(q) &= \frac{1}{6}(7P(q^7) - P(q))(343Q(q^7) - Q(q)) \\ &\quad +48z^2 x(-13 + 196x + 4459x^2 + 19208x^3). \end{aligned}$$

Applying the first part of Lemma 3.1 to the first term on the right, we complete the proof. □

Theorem 3.4.

$$\begin{aligned} Q(q) &= z\sigma(1 + 245x + 2401x^2), \\ Q(q^7) &= z\sigma(1 + 5x + x^2), \\ R(q) &= z^2(1 - 490x - 21609x^2 - 235298x^3 - 823543x^4), \\ R(q^7) &= z^2(1 + 14x + 63x^2 + 70x^3 - 7x^4). \end{aligned}$$

Proof. From the last part of Lemma 2.3 and the definitions of z and x we have

$$Q(q)^3 - R(q)^2 = 1728z^4 x, \tag{3.3}$$

$$Q(q^7)^3 - R(q^7)^2 = 1728z^4 x^7. \tag{3.4}$$

Let

$$u_1 = \frac{Q(q)}{z\sigma}, \quad u_2 = \frac{Q(q^7)}{z\sigma}, \quad u_3 = \frac{R(q)}{z^2}, \quad u_4 = \frac{R(q^7)}{z^2}.$$

Observe that when $q = 0$ we have $x = 0$, $z = 1$ and $u_1 = u_2 = u_3 = u_4 = 1$. The results of Theorem 3.3 and (3.3) and (3.4) imply

$$\begin{aligned} -u_1 + 49u_2 &= 48(1 - 49x^2), \\ bu_1 - 343bu_2 - u_3 + 343u_4 &= 48x(-13 + 196x + 4459x^2 + 19208x^3), \\ bu_1^3 - u_3^2 &= 1728x, \\ bu_2^3 - u_4^2 &= 1728x^7, \end{aligned}$$

where $b = 1 + 13x + 49x^2$. This system may be solved by the same method that was used for (2.3)–(2.6). This time, approximations as far as the quartic need to be computed. The result is

$$\begin{aligned} u_1 &= 1 + 245x + 2401x^2, \\ u_2 &= 1 + 5x + x^2, \\ u_3 &= 1 - 490x - 21609x^2 - 235298x^3 - 823543x^4, \\ u_4 &= 1 + 14x + 63x^2 + 70x^3 - 7x^4. \end{aligned}$$

It is easily checked that this is in fact a solution, and the implicit function theorem guarantees that the solution is unique. This completes the proof. □

Remark. Ramanujan gave u_3 and u_4 in factored form, namely

$$\begin{aligned} u_3 &= \left(1 - 7^2\,(5 + 2\sqrt{7})x - 7^3\,(21 + 8\sqrt{7})x^2\right) \\ &\quad \times \left(1 - 7^2\,(5 - 2\sqrt{7})x - 7^3\,(21 - 8\sqrt{7})x^2\right), \\ u_4 &= \left(1 + (7 + 2\sqrt{7})x + (21 + 8\sqrt{7})x^2\right) \\ &\quad \times \left(1 + (7 - 2\sqrt{7})x + (21 - 8\sqrt{7})x^2\right). \end{aligned}$$

Theorem 3.5.

$$\begin{aligned} j(\tau) &= (1 + 13x + 49x^2)(1 + 245x + 2401x^2)^3/x, \\ j(7\tau) &= (1 + 13x + 49x^2)(1 + 5x + x^2)^3/x^7. \end{aligned}$$

Proof. These follow from the definition of the j-function, Lemma 3.1 and Theorem 3.4. □

Theorem 3.6.

$$\sigma^2 x \frac{d}{dx}\left(\frac{\sigma^2 x}{z}\frac{dz}{dx}\right) = \frac{1}{3}\left(\frac{\sigma^2 x}{z}\frac{dz}{dx}\right)^2 - \frac{7}{3}x(3 + 28x)\sigma z.$$

Proof. Applying $q\frac{d}{dq}$ to the last result in Theorem 3.2 we get

$$\frac{d}{dx}\left(\frac{\sigma^2 x}{z}\frac{dz}{dx}\right) q\frac{dx}{dq} = \frac{7}{24} q \frac{d}{dq}\left(P(q) - P(q^7)\right).$$

Use Lemma 2.3 and the first part of Theorem 3.2 to compute the derivatives. The result is

$$\sigma^2 x \frac{d}{dx}\left(\frac{\sigma^2 x}{z}\frac{dz}{dx}\right) = \frac{7}{288}\left(P(q)^2 - 7P(q^7)^2 - Q(q) + 7Q(q^7)\right). \tag{3.5}$$

From the first part of Lemma 3.1 and the last part of Theorem 3.2 we have

$$P(q) = \sigma^2 + 4\frac{\sigma^2 x}{z}\frac{dz}{dx} \quad \text{and} \quad P(q^7) = \sigma^2 + \frac{4}{7}\frac{\sigma^2 x}{z}\frac{dz}{dx}. \tag{3.6}$$

Using these and Theorem 3.4 in (3.5) we get

$$\sigma^2 x\frac{d}{dx}\left(\frac{\sigma^2 x}{z}\frac{dz}{dx}\right)=\frac{7}{288}\left(\frac{96\sigma^4x^2}{7z^2}\left(\frac{dz}{dx}\right)^2-6\sigma^4+z\sigma(6-210x-2394x^2)\right).$$

Simplifying this, and using the second part of Lemma 3.1, we complete the proof. □

Corollary 3.7. *Let*

$$y=\frac{7x}{1+7x},\quad b=(1+13x+49x^2)^{1/3}\quad \text{and}\quad f(x)=\frac{-b^2x}{7z}\frac{dz}{dx}.$$

Then

$$\frac{d}{dx}\left(x\frac{d\sigma}{dx}\right)=2\frac{(1+16x+49x^2)}{(1+13x+49x^2)^2}\sigma, \tag{3.7}$$

$$\frac{d}{dy}\left(y(1-y)\frac{d\sigma}{dy}\right)+2\frac{(2y^2-2y-7)}{(y^2-y+7)^2}\sigma=0. \tag{3.8}$$

$$\begin{aligned} &q\frac{d}{dq}\left(P(q)-P(q^7)\right) \\ &\quad=\frac{7}{72}\left(P(q)-P(q^7)\right)^2-\frac{7}{75}\left(Q(q)-Q(q^7)\right)-\frac{8}{5}\sigma zx. \end{aligned} \tag{3.9}$$

Furthermore, we have

$$P(q) = z^{2/3}\left(b^2-28f(x)\right), \tag{3.10}$$

$$P(q^7) = z^{2/3}\left(b^2-4f(x)\right) \tag{3.11}$$

and

$$b\frac{df}{dx}=1+\frac{28}{3}x+\frac{7}{3bx}\left(f(x)\right)^2. \tag{3.12}$$

Proof. The second part of Lemma 3.1 may be used to eliminate z from the differential equation in Theorem 3.6. We suppress the details of the routine but lengthy calculation; the result is (3.7). Equation (3.8) follows immediately from (3.7) by making the change of variable from x to y.

Let us prove (3.9). From the first part of Theorem 3.2 we have

$$\sigma^2 x\frac{d}{dx}=q\frac{d}{dq}.$$

Using this together with the last part of Theorem 3.2, we see that the result in Theorem 3.6 is equivalent to

$$\frac{7}{24}q\frac{d}{dq}\left(P(q)-P(q^7)\right)=\frac{49}{1728}\left(P(q)-P(q^7)\right)^2-\frac{7}{3}x(3+28x)\sigma z. \tag{3.13}$$

Next, by Theorem 3.4 we have

$$x(3+28x)\sigma z=\frac{7}{600}\left(Q(q)-Q(q^7)\right)+\frac{\sigma zx}{5}. \tag{3.14}$$

Substituting (3.14) into (3.13) we obtain (3.9).

The second part of Lemma 3.1 may be rewritten as

$$\sigma = z^{1/3} b. \tag{3.15}$$

Using this, we see that the equations (3.10) and (3.11) are just restatements of the identities in (3.6). It remains to prove (3.12). Applying the product rule to the definition of $f(x)$, and making use of (3.15), we get

$$\begin{aligned} \frac{df}{dx} &= -\frac{1}{7}\frac{d}{dx}\left(\frac{b^2 x}{z}\frac{dz}{dx}\right) \\ &= -\frac{1}{7}\frac{d}{dx}\left(\frac{1}{z^{2/3}} \times \frac{\sigma^2 x}{z}\frac{dz}{dx}\right) \\ &= -\frac{1}{7}\left\{\frac{1}{z^{2/3}}\frac{d}{dx}\left(\frac{\sigma^2 x}{z}\frac{dz}{dx}\right) - \frac{2\sigma^2 x}{3z^{8/3}}\left(\frac{dz}{dx}\right)^2\right\}. \end{aligned}$$

Applying Theorem 3.6 and simplifying, this becomes

$$\frac{df}{dx} = \frac{\sigma^2 x}{21 z^{8/3}}\left(\frac{dz}{dx}\right)^2 + \frac{z^{1/3}}{3\sigma}(3 + 28x).$$

Finally, utilizing (3.15) to eliminate all occurrences of σ, and using the definition of $f(x)$, we complete the proof of (3.12). □

Remarks. The equations (3.7) and (3.8) are linear, and (3.8) has the additional property of being invariant when y is replaced with $1 - y$. The equation (3.8) was found by Chan and Ong [7, Theorem 2.4], and (3.9)–(3.12) were given by Raghavan and Rangachari [16]. Equations (3.7) and (3.8) are differential equations for σ, while (3.12) should be viewed as a differential equation for z.

4. Application to convolution sums

Let $\sigma_j(n)$ denote the sum of the j-th powers of the divisors of n. The convolution sum

$$W_k(n) = \sum_{1 \le m < n/k} \sigma_1(m)\sigma_1(n - km)$$

has been evaluated for $k = 1, \ldots, 14$, 16, 18, 23 and 24. See the work of K. S. Williams and his coauthors, for example [1], and the papers [6] and [21]. Corollaries 2.9 and 3.7 readily yield results for $W_5(n)$ and $W_7(n)$.

Corollary 4.1. *For $n = 1, 2, 3, \ldots$, let $c_5(n)$ and $c_7(n)$ be the numbers defined by*

$$\begin{aligned} \sum_{n=1}^{\infty} c_5(n) q^n &= q(q)_\infty^4 (q^5)_\infty^4, \\ \sum_{n=1}^{\infty} c_7(n) q^n &= q(q)_\infty^3 (q^7)_\infty^3 \sigma(q). \end{aligned}$$

Then

$$\begin{aligned} W_5(n) &= \frac{5}{312}\left(\sigma_3(n)+25\sigma_3\left(\frac{n}{5}\right)\right) \\ &\quad+\left(\frac{1}{24}-\frac{n}{20}\right)\sigma_1(n)+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma_1\left(\frac{n}{5}\right)-\frac{1}{130}c_5(n), \\ W_7(n) &= \frac{1}{120}\left(\sigma_3(n)+49\sigma_3\left(\frac{n}{7}\right)\right) \\ &\quad+\left(\frac{1}{24}-\frac{n}{28}\right)\sigma_1(n)+\left(\frac{1}{24}-\frac{n}{4}\right)\sigma_1\left(\frac{n}{7}\right)-\frac{1}{70}c_7(n). \end{aligned}$$

Proof. For $j = 1$, 5 or 7, let $P_j = P(q^j)$ and $Q_j = Q(q^j)$. By Lemma 2.3 we have

$$\begin{aligned} (P_1-P_5)^2 &= P_1^2+P_5^2-2P_1P_5 \\ &= 12q\frac{dP_1}{dq}+\frac{12}{5}\frac{dP_5}{dq}+Q_1+Q_5-2P_1P_5. \end{aligned}$$

Substituting this into the first term on the right hand side of (2.13) and solving the resulting equation for P_1P_5, we obtain

(4.1)

$$P_1P_5 = \frac{6}{5}\left(q\frac{dP_1}{dq}+5q\frac{dP_5}{dq}\right)+\frac{1}{26}\left(Q_1+25Q_5\right)-\frac{288}{65}q(q)_\infty^4(q^5)_\infty^4.$$

Similarly, from Corollary 3.7 we obtain

(4.2)

$$P_1P_7 = \frac{6}{7}\left(q\frac{dP_1}{dq}+7q\frac{dP_5}{dq}\right)+\frac{1}{50}\left(Q_1+49Q_7\right)-\frac{288}{35}q(q)_\infty^3(q^7)_\infty^3\sigma(q).$$

Now equate coefficients of q^n on both sides of the identities (4.1) and (4.2) and make of the expansions

$$P_j = 1-24\sum_{k=1}^{\infty}\sigma_1(k)q^{jk} \quad \text{and} \quad Q_j = 1+240\sum_{k=1}^{\infty}\sigma_3(k)q^{jk},$$

to complete the proof. □

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